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The effect of a parallel magnetic field on the stability of free boundary-layer type flows of low magnetic Reynolds number

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Stability to infinitesimal disturbances, when a parallel magnetic field is imposed, is investigated for the free boundary-layer type flows, of low magnetic Reynolds number, between two unbounded parallel streams of a viscous, incompressible, electrically conducting fluid. Neutral stability curves are calculated for small wave-number making use of the limiting profile: previous results by another author are found to be incomplete. A qualitative neutral stability picture is conjectured for other values of the wave-number and, granted a certain part of this conjecture, the conclusion is that the critical Reynolds number remains zero until the parameter Q/R exceeds the value $(Q/R)_{\text{crit}} \doteqdot 0{\cdot}0233$. It is suggested that a sufficiently strong magnetic field can stabilize a flow of any finite Reynolds number.

1. Introduction

Beginning with an investigation by Lessen (1950), the hydrodynamic stability of free boundary-layer profiles has been studied by several authors. Out of their work has emerged the information (see Tatsumi & Gotoh 1960) that the critical Reynolds number of these profiles is zero, and that the ratio R/α of the Reynolds number to the wave-number of the neutrally stable disturbance tends to the constant $1/4\sqrt{3}$ as $\alpha \rightarrow 0$.

Our main concern here is to discover how a uniform magnetic field in the flow direction affects this zero critical Reynolds number when the flow is that of an electrically conducting fluid. An investigation of this has previously been carried out by Gotoh (1961), but owing to very elaborate manipulations involved in his analysis Gotoh failed to elicit an important piece of information, namely another existing branch of the neutral stability curve. Also, he considered the effects of two-dimensional disturbances only, which at the time of publication of his paper were incorrectly thought to be the most destabilizing. Thus Gotoh's work uncovered only partially the effects of the magnetic field and he came to incorrect conclusions regarding the distribution of the stable/unstable regions.

Since in the region of real interest the wave-number α is small we thought it most appropriate to use an approach suggested by Drazin (1961). As a first method of attack we found it to be most rewarding, for it simplified the analysis enormously, it yielded information previously missed, and it suggested lines of inquiry which might be pursued using more sophisticated analysis. Drazin, in the paper we refer to, interpreted the use of certain simple, and sometimes physically inconsistent, velocity profiles as 'limiting profiles' which may legitimately be used for finding stability characteristics in the limit of small wave-number. Mathematically speaking, he proved that the same eigenvalues c are obtained when for a given velocity profile the wave-number tends to zero or alternatively when the limiting profile is considered, the limiting profile being the one obtained by letting the length scale tend to zero. Following Drazin we have used the limiting profile for the free boundary-layer flows, which is the Helmholtz profile, to discover their stability characteristics in the region of small wave-numbers.

2. The mathematical formulation

Let the steady state in the flow of an incompressible fluid of uniform conductivity σ, density ρ, kinematic viscosity ν and magnetic permeability μ, characterized by a velocity $\{\overline{U}(x_2), 0, 0\}$ and a uniform magnetic field $\{H, 0, 0\}$, be disturbed by a small velocity and a small magnetic field, respectively of the form

$$\left.\begin{aligned}\mathbf{v} &= \{v_1(x_2), v_2(x_2), v_3(x_2)\}\\ \mathbf{h} &= \{h_1(x_2), h_2(x_2), h_3(x_2)\}\end{aligned}\right\} \exp\{i(\alpha_1 x_1 + \alpha_3 x_3) + \omega t\}.$$

Then it can be shown (see Stuart 1954) that the equation governing the disturbance, when the magnetic Reynolds number is assumed to be small, takes the form

$$(U-c)(\phi''-\beta^2\phi) - U''\phi + i\alpha Q\phi = \frac{1}{i\alpha R}(\phi'''' - 2\beta^2\phi'' + \beta^4\phi). \tag{2.1}$$

Dashes in the above denote differentiation with respect to y which along with the other dimensionless quantities has been obtained by writing

$$\left.\begin{aligned} y &= x_2/L, \quad U(y) = \overline{U}(x_2)/U_0, \quad \phi = v_2/U_0,\\ \alpha &= \alpha_1 L, \quad c = i\omega/\alpha_1 U_0,\\ \beta &= L\sqrt{(\alpha_1^2+\alpha_3^2)}, \quad R = U_0 L/\nu,\\ R_m &= 4\pi\mu\sigma U_0 L, \quad A = \sqrt{(\mu/4\pi\rho)}\,H/U_0, \quad Q = A^2 R_m.\end{aligned}\right\} \tag{2.2}$$

In (2.2) U_0 and L denote suitably chosen reference quantities for the velocity and the length scales respectively.

By writing

$$\alpha R = \beta\overline{R} \quad \text{and} \quad \alpha Q = \beta\overline{Q},$$

or alternatively

$$\overline{R} = R\cos\theta \quad \text{and} \quad \overline{Q} = Q\cos\theta, \tag{2.3}$$

where $\theta = \tan^{-1}(\alpha_3/\alpha_1)$, it is easily seen that the three-dimensional problem associated with (2.1) is immediately reduced to an equivalent two-dimensional problem (see Stuart 1954). In what follows below we shall first consider the effects of two-dimensional disturbances. Later in §5 when we come to consider the effects of three-dimensional disturbances we shall replace our R and Q by $R\cos\theta$ and $Q\cos\theta$ respectively, as dictated by the transformations (2.3).

3. The two-dimensional disturbances

Consider for the moment two-dimensional disturbances only in the basic flow

$$U = y/|y| \quad (-\infty < y < \infty). \tag{3.1}$$

The most general solution of (2.1) (with $\beta = \alpha$) satisfying the boundary condition that the disturbances should vanish at infinity is of the form

$$\phi = \begin{cases} A_1 e^{-m_1 y} + A_2 e^{-m_2 y} & (y > 0), \\ B_1 e^{n_1 y} + B_2 e^{n_2 y} & (y < 0), \end{cases} \tag{3.2}$$

where A_1, A_2, B_1, B_2 are constants and

$$\left.\begin{aligned} m_1 &= \left[\alpha^2 + \frac{i\alpha R}{2}(1-c) - \frac{i}{2}\{\alpha^2R^2(1-c)^2 + 4\alpha^2 QR\}^{\frac{1}{2}}\right]^{\frac{1}{2}}, \\ m_2 &= \left[\alpha^2 + \frac{i\alpha R}{2}(1-c) + \frac{i}{2}\{\alpha^2R^2(1-c)^2 + 4\alpha^2 QR\}^{\frac{1}{2}}\right]^{\frac{1}{2}}, \\ n_1 &= \left[\alpha^2 - \frac{i\alpha R}{2}(1+c) + \frac{i}{2}\{\alpha^2R^2(1+c)^2 + 4\alpha^2 QR\}^{\frac{1}{2}}\right]^{\frac{1}{2}}, \\ n_2 &= \left[\alpha^2 - \frac{i\alpha R}{2}(1+c) - \frac{i}{2}\{\alpha^2R^2(1+c)^2 + 4\alpha^2 QR\}^{\frac{1}{2}}\right]^{\frac{1}{2}}. \end{aligned}\right\} \tag{3.3}$$

(The square roots are taken to make the real part positive.)

The boundary conditions to be satisfied by ϕ and its derivatives on the surface of transition $y = 0$ are obtained by integrating (2.1) across this surface, and are

$$\left.\begin{aligned} [\phi] = 0, \quad [\phi'] = 0, \\ [\phi'' - i\alpha R(U-c)\phi] = 0, \quad [\phi''' - i\alpha R\{(U-c)\phi' - U'\phi\}] = 0, \end{aligned}\right\} \tag{3.4}$$

where $[\gamma]$ denotes the jump in any quantity γ in crossing $y = 0$. [*Note.* The problem of deciding on the boundary conditions when a physically inconsistent velocity profile is used in a method of approximation contains pitfalls. The reason for this is that the usual arguments about the continuity of stress, etc. cannot be applied. The author has discussed this, deriving two alternative sets of boundary condition for a problem, in the work referred to at the end as Abas (1967). Conditions (3.4) will also become clear if the reader will refer to the paper by Drazin (1961), where, by writing down the successive integrals, he derives the boundary conditions for the Orr–Sommerfeld equation.]

The boundary conditions (3.4) at $y = 0$ give four homogeneous linear equations in A_1, A_2, B_1 and B_2. A non-trivial solution exists if and only if the eliminant of these equations is zero, i.e. if

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -m_1 & -m_2 & n_1 & n_2 \\ m_1^2 + i\alpha R & m_2^2 + i\alpha R & n_1^2 - i\alpha R & n_2^2 - i\alpha R \\ -m_1^3 + i\alpha R m_1 & -m_2^3 + i\alpha R m_2 & n_1^3 + i\alpha R n_1 & n_2^3 + i\alpha R n_2 \end{vmatrix} = 0. \tag{3.5}$$

Equation (3.5) is the eigenvalue relationship which determines c for given values of α, R and Q.

Taking out the factors (m_2-m_1) and (n_2-n_1), (3.5) becomes

$$\begin{matrix}(m_2-m_1)\\(n_2-n_1)\end{matrix}\begin{vmatrix}1 & -(m_2+n_2) & 1\\ -(m_1+m_2) & m_2^2-n_2^2+2i\alpha R & n_1+n_2\\ m_1^2+m_2^2+m_1m_2-i\alpha R & -(m_2^3+n_2^3)+i\alpha R(m_2-n_2) & n_1^2+n_2^2+n_1n_2+i\alpha R\end{vmatrix} = 0. \quad (3.6)$$

Now, from (2.1) making use of the fact that U is an odd function of y, it is fairly easy to show that the equation satisfied by $\bar{\phi}(-y)$, where a bar denotes a complex conjugate, is the same as (2.1) except that $-c$ is replaced by $\bar{c}$. We shall assume that for given values of α, R and Q the neutrally stable waves are characterized by a unique eigenvalue c. Then from the above it follows that $c=-\bar{c}$, thus giving $R_e\, c=0$. [In the ordinary non-conducting case $c_r=0$ was shown to be the only root by Drazin (1961) by the explicit solution of the equivalent eigenvalue relationship. Hence another reason why $c_r=0$, in our case also, is the continuity of c_r.]

The curve of neutral stability is therefore characterized by the condition $c=0$ and at this stage we shall restrict our analysis to the neutral case.

We note then, from (3.3), that when $c=0$

$$n_1=\bar{m}_1 \quad\text{and}\quad n_2=\bar{m}_2, \quad (3.7)$$

and also

$$m_1^2+m_2^2=2\alpha^2+i\alpha R,\quad n_1^2+n_2^2=2\alpha^2-i\alpha R,\quad m_2^2-n_2^2=i\alpha R(1+T), \quad (3.8)$$

where

$$T=\left(1+\frac{4Q}{R}\right)^{\frac{1}{2}}. \quad (3.9)$$

Relationships (3.8) can now be used to simplify (3.6) resulting in

$$\begin{matrix}(m_2-m_1)\\(n_2-n_1)\\(m_2+n_2)\end{matrix}\begin{vmatrix}m_1+m_2+n_1+n_2 & a(m_2-n_2)+(n_1+n_2)\\ n_1n_2-m_1m_2 & (a-1)\alpha^2+(2-a)m_2n_2+n_1n_2\end{vmatrix}=0, \quad (3.10)$$

where we have written

$$a=\frac{3+T}{1+T}. \quad (3.11)$$

The only roots relevant to the problem are given by the factor in the determinant and expanding it we get

$$(m_1+n_1)\left[(a-1)\alpha^2+(3-2a)m_2n_2\right]+(m_2+n_2)\left[(a-1)\alpha^2+m_1n_1+(2-a)m_2n_2\right] + a[m_1m_2^2+n_1n_2^2]=0. \quad (3.12)$$

Dividing by α^3 and writing

$$\delta=R/\alpha, \quad (3.13)$$

$$\left.\begin{aligned} m_1' &= \left[1-\left(\frac{T-1}{2}\right)i\delta\right]^{\frac{1}{2}}, & m_2' &= \left[1+\left(\frac{T+1}{2}\right)i\delta\right]^{\frac{1}{2}},\\ n_1' &= \bar{m}_1', & n_2' &= \bar{m}_2', \end{aligned}\right\} \quad (3.14)$$

the equation becomes

$$(m_1'+n_1')[(a-1)+(3-2a)|m_2'|^2]+(m_2'+n_2')[(a-1)$$
$$+|m_1'|^2+(2-a)|m_2'|^2]+a[m_1'm_2'^2+n_1'n_2'^2] = 0. \quad (3.15)$$

All the terms occurring in (3.15) are real and, if we write

$$m_1' = r_1 e^{i\gamma_1}, \quad m_2' = r_2 e^{i\gamma_2},$$

then (3.15) becomes

$$r_1 \cos\gamma_1[(a-1)+(3-2a)r_2^2]+r_2\cos\gamma_2[(a-1)+r_1^2+(2-a)r_2^2]$$
$$+ar_1r_2\cos(\gamma_1+2\gamma_2). \quad (3.16)$$

Solving for r_1, r_2, γ_1 and γ_2 in terms of δ and T and then substituting in (3.16) gives the equation

$$5+T+(T-3)\sqrt{q}+\left(\frac{\sqrt{q}+1}{\sqrt{p}+1}\right)^{\frac{1}{2}}\{2+(T+1)\sqrt{p}+(r-1)\sqrt{q}\}$$
$$+(T+1)(T+3)\frac{\delta}{2}\left(\frac{\sqrt{p}-1}{\sqrt{p}+1}\right)^{\frac{1}{2}} = 0, \quad (3.17)$$

where
$$p = 1+\left(\frac{T-1}{2}\right)^2\delta^2, \quad q = 1+\left(\frac{T+1}{2}\right)^2\delta^2. \quad (3.18)$$

We immediately note from (3.17) that, if $T \geqslant 3$, then, since all the terms are positive, there can be no real values of δ satisfying the equation. This means that $c = 0$ is not an eigenvalue if $T > T_{\text{crit}}$, where

$$1 < T_{\text{crit}} < 3. \quad (3.19)$$

Equation (3.17) was solved numerically and gave

$$T_{\text{crit}} \doteqdot 1{\cdot}0456. \quad (3.20)$$

Next we show that the left-hand side of (3.17) has a different behaviour in the case $T = 1$ (i.e. zero magnetic field) as compared with the case $T > 1$.

Let us write (3.17) as $F(\delta, T) = 0$. Then, when $T = 1$

$$F(\delta,1) = 6+2\sqrt{2}(1+\sqrt{(1+\delta^2)})^{\frac{1}{2}}-2\sqrt{(1+\delta^2)}. \quad (3.21)$$

For large values of δ
$$F(\delta,1) \sim 2\sqrt{2}\,\delta^{\frac{1}{2}}-2\delta,$$

and is seen to be negative. When $\delta = 0$, F has the value 8, so that the equation $F(\delta,1) = 0$ is deemed to have an odd number of real positive roots. In fact there is only one root $\delta = 4\sqrt{3}$ as is easily found from (3.21) by squaring twice to eliminate radicals. This is as found by Esch (1957), Tatsumi & Gotoh (1960) and Drazin (1961).

On the other hand, when $T > 1$, then for large values of δ

$$F(\delta,T) \sim (T+1)\{T+(T^2-1)^{\frac{1}{2}}\}\delta,$$

and is seen to be positive. When $\delta = 0$, F has the value $4(T+1)$ which is also positive, so that if real positive roots exist they must be even in number.

Equation (3.17) was solved numerically using a computer and was found to have two roots when $1 < T < T_{\text{crit}}$. The values obtained are shown graphically in figure 1.

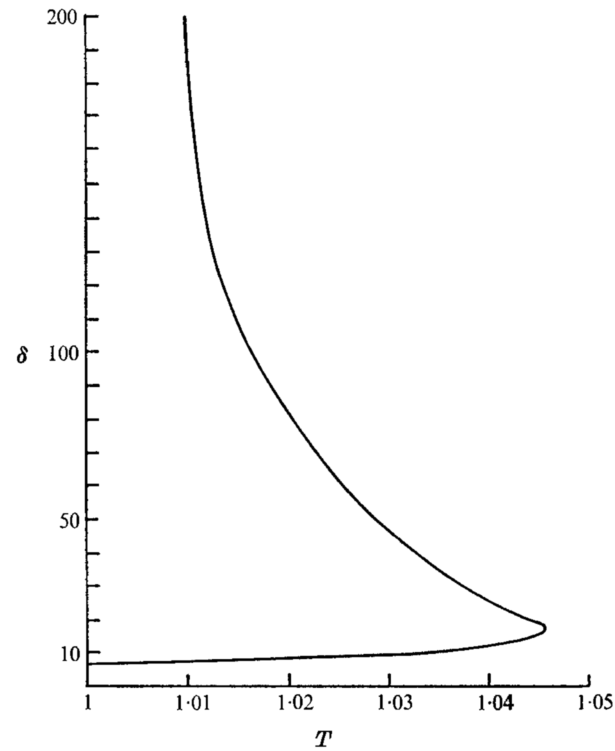

FIGURE 1. Eigenvalues of neutral stability.

4. Conclusions from the results of §3

We found that when $Q/R \ll 1$ a perturbation from the solution $\delta = 4\sqrt{3}$ when $Q/R = 0$ in the equation (3.17) gave the result

$$\delta = 4\sqrt{3}\left(1+\frac{13Q}{R}\right),$$

which is the same as

$$\frac{R}{\alpha} = 4\sqrt{3}\left(1+\frac{13A^2R_m}{R}\right) \tag{4.1}$$

found by Gotoh (1961, no. 54, p. 567). Gotoh did not however consider the possibility of another root which for $Q/R \ll 1$ is extremely large. Owing to this omission his speculations on the distributions of the stable and the unstable regions, and his conclusion that the magnetic field, to begin with, has no influence on very long wave disturbances, are incorrect.

From our results, the curves of neutral stability for small values of α are as shown in figure 2. The existence of two values of R/α for the same value of Q/R implies that the unstable region, if it occurs for a value of Q/R, is contained within the stable region.

The unstable region shrinks as Q/R increases and disappears when it exceeds $(Q/R)_{\text{crit}}$, where

$$(Q/R)_{\text{crit}} \doteqdot 0{\cdot}0233. \tag{4.2}$$

Thus, contrary to the conclusion of Gotoh (1961), we find that the stabilizing influence of the magnetic field is felt immediately on disturbances of very long wavelengths and also on ones of shorter wavelengths. Also, we note that, although the unstable region shrinks with an increasing magnetic field, the minimum critical Reynolds number (we suspect the existence of a maximum critical Reynolds number, see § 6) remains zero until Q/R reaches its critical value when the flow is stabilized.

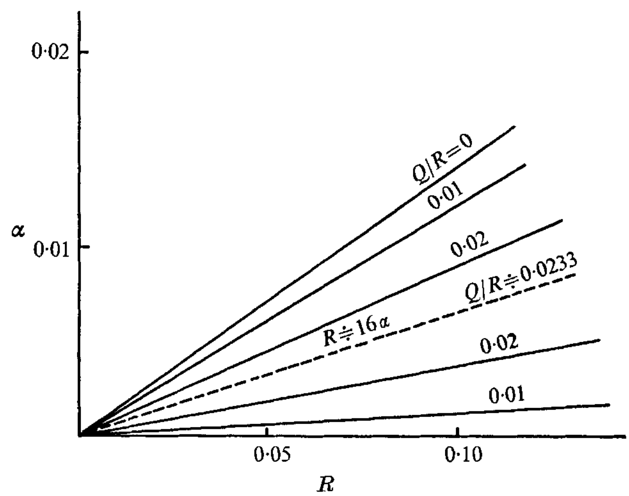

FIGURE 2. Curves of neutral stability for two-dimensional disturbances.

5. Modification of results for three-dimensional disturbances

So far we have considered only two-dimensional disturbances in the basic flow. If we now replace Q by $Q\cos\theta$ and R by $R\cos\theta$, θ being the angle at which a three-dimensional disturbance propagates with respect to the basic flow, then the same value of T increases δ by the factor $1/\cos\theta$. In figure 3 we have plotted the neutral curves for $Q/R = 0{\cdot}02$ and for $\theta = 0°$ and $60°$. For $\theta > 0$ the neutral curves are always pulled towards the R-axis though the rotation produced in the two branches is not uniform. The unstable region still disappears for the same critical value of Q/R. We conclude, therefore, that for small values of the wave-number the free boundary-layer type flows are stable if

$$Q/R > (Q/R)_{\text{crit}} \doteqdot 0{\cdot}0233.$$

6. Stability characteristics for other values of α

Although our neutral stability curves are applicable for small values of α only, we can use this fractional information to draw a highly plausible qualitative picture in the rest of the (α, R)-plane.

From comparison with related work on flows with solid boundaries where complete neutral curves have been worked out (see, for example, Abas 1968), it seems most likely that for $0 < Q/R < (Q/R)_{crit}$ the two-dimensional analysis

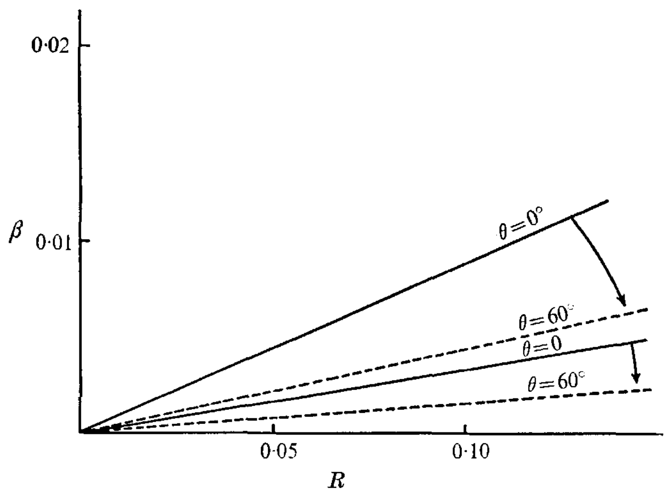

FIGURE 3. Curves of neutral stability for $Q/R = 0{\cdot}02$ and $\theta = 0°, 60°$. β is the wave-number in the θ-direction.

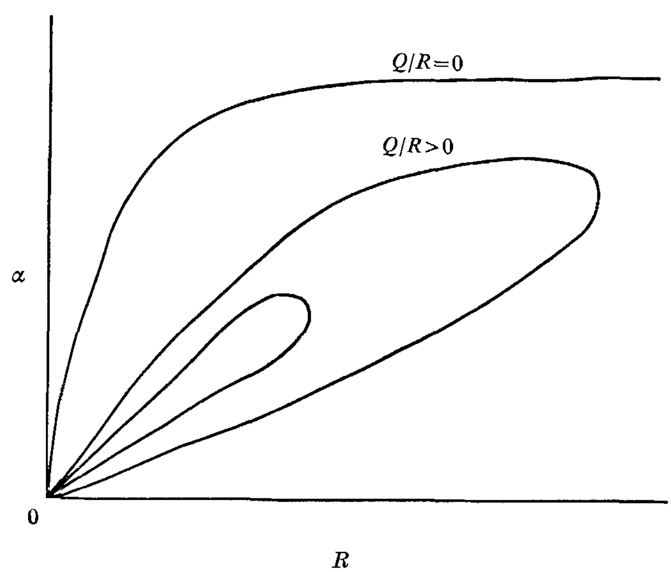

FIGURE 4. Conjectured form of the neutral stability curves for $Q/R > 0$.

yields neutral curves that are closed, as shown in figure 4. It should of course be clear that as far as the action of viscosity is concerned the mechanism of instability in the case of semi-bounded or bounded flows is very different from that of an unbounded flow, hence the reason for small αR in the stability characteristics of the latter. However, it would be expected that the action of the magnetic field on the stability characteristics of the two would be similar. It is therefore most probable that in our case the magnetic field yields a zero minimum critical Reynolds number but there is also a maximum critical Reynolds number beyond

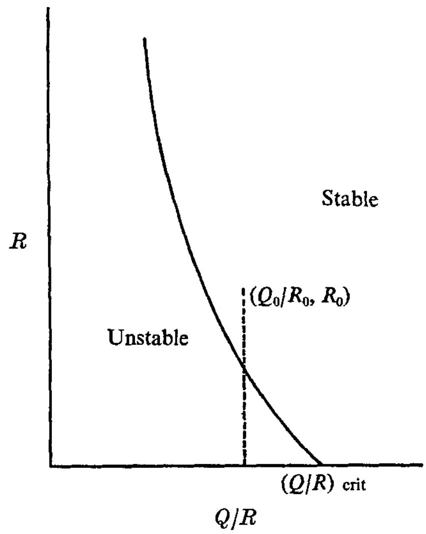

FIGURE 5. Stability limits for two-dimensional disturbances.

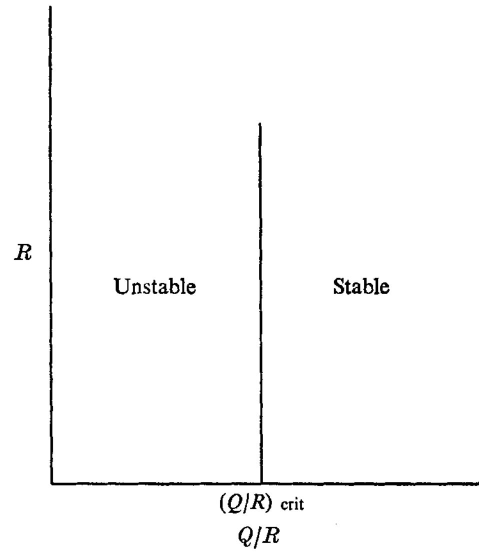

FIGURE 6. Stability limits for three-dimensional disturbances.

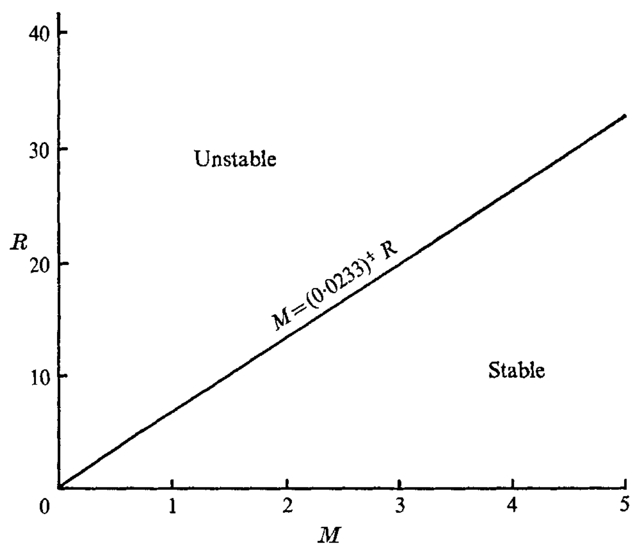

FIGURE 7. The critical Reynolds number in terms of M.

which the flow is stable to all two-dimensional disturbances. The value of the maximum critical Reynolds number, unlike the minimum critical Reynolds number, will most probably be different for different profiles. The neutral curves shrink in area as Q/R increases from zero and disappear when Q/R exceeds $(Q/R)_{crit}$.

If the above conjecture is correct then the geometry of the stable/unstable regions in the $(Q/R, R)$-plane is as shown in figure 5.

When we come to consider three-dimensional disturbances then, as before, a typical point $(Q_0/R_0, R_0)$ will transform into the point $(Q_0/R_0, R_0 \cos\theta)$. The point will therefore lie on a line parallel to R-axis such as the dotted line in figure 5. It is easy to see that the point $(Q_0/R_0, R_0)$ belongs to the unstable region for some value of θ if and only if $Q_0/R_0 < (Q/R)_{\text{crit}}$. If therefore we do not restrict ourselves to the two-dimensional disturbances then the stable and the unstable regions are situated as in figure 6.

We have also shown the unstable and the stable regions in the (M, R)-plane in figure 7, where M is the parameter defined by

$$M^2 = QR.$$

Note that M is proportional to the magnetic field H and, unlike Q, is independent of the velocity term. The curve in the (M, R)-plane is therefore more immediately instructive to the effect of the applied magnetic field.

7. Closing comments

Although we conjecture that the neutral stability curves when $Q/R > 0$ are closed curves our deduction that the flow is stable if Q/R exceeds $(Q/R)_{\text{crit}}$ is independent of this property. Even if the two branches go to infinity, so long as the unstable region shrinks in area and disappears when Q/R exceeds $(Q/R)_{\text{cri}}$) the relationship in the $(Q/R, R)$-plane between regions stable and unstable to two-dimensional disturbances will be as shown in figure 6. For three-dimensional disturbances the point $(Q_0/R_0, R_0)$ transforms into the point $(Q_0/R_0, R_0 \cos\theta)$ and therefore, since it moves parallel to the R-axis, cannot cross from one region into another. Thus the flow is still stable if $Q/R > (Q/R)_{\text{crit}}$.

We conclude that providing the neutral curves completely disappear for $(Q/R) > (Q/R)_{\text{crit}}$ all free boundary-layer type flows of low magnetic Reynolds number and of any fixed Reynolds number can be stabilized by imposing a sufficiently large parallel magnetic field. It is probably worth emphasizing that by a sufficiently large field we mean a field such that the parameter $Q/R\,(\propto H^2/U^2)$ exceeds its critical value and that this critical value ($\doteqdot 0{\cdot}0233$) is universally valid for all free boundary-layer flows. It should also be clear (see figure 7) that for a given fixed magnetic field the flow is never stable for all values of the Reynolds number. This result is the same as that obtained by Hunt (1966 for the case of flows contained within solid boundaries.

Finally we should like to remark that the closure of the neutral stability curves, reflecting the stabilizing effect of a parallel field on the larger as well as the shorter unstable wavelengths, seems to be a common feature in the stability characteristics calculated so far. There is, however, as yet no mathematical analysis which has shown this to be a universal feature and an attack on this seems to be now due.

The work reported here was carried out in 1965 when the author was a research student at University College London and was included in a part of his Ph.D.

thesis submitted to the University of London in March 1967. The author wishes to thank Dr D. H. Michael who throughout this piece of research gave helpful guidance and from whose remarks on Esch's (1957) boundary condition the author's interest in the stability of free boundary-layer flows was aroused.

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